WAVE PROPAGATION PROBLEMS THROUGH MULTI-LAYERED MEDIA

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Abstract: *The mathematical description of wave boundary value problems contains functional equations. They can be: matrix differential, and/or partial differential, and/or integral, and/or integral-differential equations. They are associated with integral and/or integral-differential boundary conditions. The description can be algebraized by a suitable integral transformation. The derivatives and integrals in the functional equations with respect to the time variable "t" are transformed into polynomial functions of the complex variable "s". This transformation is performed according to the image acquisition rules. The general scheme of the solution can be visualized by building an ORIENTED GRAPH of the task - BLOCK DIAGRAM. This graph represents the core of the structural solution. From it, the general complex transfer function of the task can easily be determined ANALYTICALLY. This function can be multiplied in the complex domain by the representation of external loadings. As a result, the image of the sought solution (integral) will be obtained. From this image, by inverse integral transformation, the desired solution in the time domain can be obtained. The kernel of the structural solution (DIRECTED GRAPH) can also be created in the time domain without looking for a transfer function. In this case, an equivalent solution (integral) of the wave boundary value problem is obtained. The solving system of equations in this time domain is DIFFERENTIAL. Efforts to ANALYTICALLY obtain the general transfer function of the task in the complex area depend not so much on the type of derivatives and/or integrals over the time variable "t" involved in the initial mathematical description, as on the STRUCTURE AND TYPE of the considered limited or unlimited spaces (areas) and from the structure and type of their BOUNDARIES [10]. The sought integrals in the space L_2 can be obtained ANALYTICALLY or NUMERICALLY from the solving algebraic system of equations with considered boundary and initial conditions.*

РАЗПРОСТРАНЕНИЕ НА ВЪЛНИ В МНОГОСЛОЙНИ СРЕДИ

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Резюме: *Математическото описание на вълнови гранични задачи съдържа функционални уравнения. Те могат да бъдат: матрични диференциални, и/или частни диференциални, и/или интегрални, и/или интегродиференциални уравнения. Свързани са с интегрални и/или интегродиференциални гранични условия. Описанието може да се алгебризира чрез подходящо интегрално преобразование. Производните и интегралите във функционалните уравнения по отношение на времевата променлива „t" се трансформират в полиномиални функции на комплексната*

променлива “s”. *Тази трансформация се извършва съгласно правилата за получаване на изображения. Общата схема на решението може да се визуализира чрез построяването на ОРИЕНТИРАН ГРАФ на задачата - БЛОКОВА СХЕМА. Този граф представлява ядро на структурното решение. От него лесно може да се определи АНАЛИТИЧНО общата комплексна предавателна функция на задачата. Тази функция може да бъде умножена в комплексната област по изображението на външните въздействия. В резултат ще се получи изображението на търсеното решение (интеграл). От това изображение чрез обратно интегрално преобразуване може да се получи самото търсено решение във времевата област. Ядрото на структурното решение (ОРИЕНТИРАН ГРАФ) може да се създаде и във временната област, без да се търси предавателна функция. В този случаи се получава еквивалентно решение (интеграл) на вълновата гранична задача. Разрешаващата система уравненя в тази временна област е ДИФЕРЕНЧНА. Усилията за АНАЛИТИЧНО получаване на общата предавателна функция на задачата в комплексноста област, зависят не толкова от вида на участващите в началното математическо описание производни и/или интеграли по времевата променлива „t”, колкото от СТРУКТУРАТА И ВИДА на разглежданите ограничени или неограничени пространства (области) и от структурата и вида на техните ГРАНИЦИ [10]. Търсените интеграли в пространството L_2 могат да се получат АНАЛИТИЧНО или ЧИСЛЕНО от разрешаващата алгебрична система уравнения с отчетени гранични и начални условия.*

1. SH (polarized in horizontal plane) wave propagation through multi layered media. The wave propagation process, at the direction of the axis *x* perpendicular to the investigated multilayered media (Fig. 1), could be described by the following equation:

(1) $$\frac{\partial^2 w(x,t)}{\partial t^2} - V_{SH}^2 \frac{\partial^2 w(x,t)}{\partial x^2} = 0 ,$$

където $V_{SH} = \sqrt{\frac{\mu}{\rho}}$ is the wave propagation velocity for the shear waves SH, μ=const is the Lame coefficient, ρ=constant represents mass density. The function *w(x,t)* is the anti plane *(X, Y)* component of the displacement vector in the direction parallel to the axis *Z*. On the boundary between the two neighbouring layers “i” and “i+1” the corresponding boundary conditions are satisfied. These boundary conditions represent that the unknown displacements and forces (stresses) are continuous:

(2) $$w(x,t)_{boundary}^{i} = w(x,t)_{boundary}^{i+1} \quad ,$$

(3) $$P_{boundary}^{i} = \left(\sigma_{ij} n_j\right)_{boundary}^{i} = -P_{boundary}^{i+1} = -\left(\sigma_{ij} n_j\right)_{boundary}^{i+1} ,$$

where Pi is the boundary force vector of the corresponding layer, “σ_{ij}” is the stress tensor, and “n_j” is the corresponding normal vector. The initial conditions with respect to the displacement and to the first difference are homogeneous. The both functions depend on the spatial variable “x” and time argument “t” in the initial moment “t=0”:

(4) $$w(x,t)\,|_{t=0} = 0, \qquad \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0.$$

2. For the DIRECT problem (identifying the free surface signal from a set signal in the bedrock) the displacement boundary conditions are set as a known function of the time variable “t”:

(5) $$w(0,t) = X_b\ (t) .$$

The boundary conditions in terms of forces on the free surface are homogeneous for both the DIRECT and the INVERSE (by a given signal on the free surface to identify what the signal was in the bed rock) problems are HOMOGENEOUS:

(6) $$P^{surface} = \left(\sigma_{ij} n_j\right)^{surface} = 0.$$

3. On the other hand, for the INVERSE problem on the free surface, the displacement boundary condition is given in the mode of the known time function “t”:

(7) $$w(l,t) = X_s\ (t).$$

4. Structural mathematical model of the multi layered structure. The structural model of the multilayer media is shown in Fig. 1. The SH Wave Propagation Reflect – Pass Perpendicular Process is illustrated on the Fig.1 a. The Block - Diagram Model of the media under investigation is shown in Fig.1 b. The Flow Graph of the system signals is shown in the Fig.1 c.

The above formulated SH wave boundary condition problem (1)–(7), could be solved by a system of differential equations, initial and boundary conditions. This differential system consists of following elements:

- n equations in the mode (1), one differential equations for each layer, because the velocity function $V(x)$ depending on spatial co-ordinate x is discontinued and is of a terrace-like type;
- 2(n - 1) boundary conditions in the mode (2), (3);
- surface boundary condition in the mode (6);
- initial conditions in the mode (4),
- and either of boundary conditions (cinematic excitation) (5) for the DIRECT problem or (7) for the INVERSE problem.

The above formulated wave boundary problem (1)–(7) can be transformed in the complex domain. The solution of the investigated problem (1)–(7) in the complex domain could be obtained by solving the following algebraic system of equations, which could be solved analitically (analitical solution of the DIRECT and INVERSE problems) as follows:

$$\mathbf{X}_1(\mathbf{s}) = \mathbf{W}_1(\mathbf{s})\left[(\mathbf{1}+\beta_1(\mathbf{s}))\mathbf{X_b}(\mathbf{s}) - \beta_1(\mathbf{s})\mathbf{X}_1^*(\mathbf{s})\right],$$

$$* * * \quad * * * \quad * * *$$

$$\mathbf{X_i}(\mathbf{s}) = \mathbf{W_i}(\mathbf{s})\left[(\mathbf{1}+\beta_\mathbf{i}(\mathbf{s}))\mathbf{X_{i-1}}(\mathbf{s}) - \beta_\mathbf{i}(\mathbf{s})\mathbf{X_i^*}(\mathbf{s})\right], \tag{8}$$

$$* * * \quad * * * \quad * * *$$

$$\mathbf{X_i^*}(\mathbf{s}) = \mathbf{W_i}(\mathbf{s})\left[\beta_\mathbf{i+1}(\mathbf{s})\mathbf{X_i}(\mathbf{s}) + (\mathbf{1}-\beta_\mathbf{i+1}(\mathbf{s}))\mathbf{X_{i+1}^*}(\mathbf{s})\right],$$

$$* * * \quad * * * \quad * * *$$

$$\mathbf{X_n^*}(\mathbf{s}) = \mathbf{W_n}(\mathbf{s})\ \mathbf{X_n}(\mathbf{s})\ .$$

The unknown variables in the system (8) (X_1, X_2,...,X_i.,...X_n, X^*_1, X^*_2,..., X^*_i,..., X^*_n) represents the displacements, velocities or accelerations of the media particles under investigation. The coefficients β=β(s)= Re β (s)+ j Im β (s) in the system (8) are reflection and refraction layer ratios (see Fig. 1 b, Fig. 1 c). They are known complex functions of the parameter of integral transformation “s”. The function matrix of the system (8) is asymmetric. Based on this fact, the common transfer function of the problem Ψ(s) could be obtained by recurrent elimination of the system parameters. This function physically represents the quotient between images of input and output signals of the geological structure under investigation:

$$\Psi(\mathbf{s}) = \frac{\left\{\mathbf{X_{output}}(\mathbf{s})\right\}}{\left\{\mathbf{X_{input}}(\mathbf{s})\right\}} \tag{9}$$

Substituting the analytical complex parameter “s” by the numerical imaginary parameter “jω” into system of equations (8), it is possible to calculate numerically the formulated DIRECT and INVERSE problems by means of Fourier integral transformation (numerical solution of the DIRECT and INVERSE problems).

5. Quadruple symmetric real functions - Fig.2. The coefficients β=β(jϖ)=Reβ(ϖ)+ jImβ(ϖ) in the system (8) are reflection and refraction layer ratios according to the Willebrord Snellius (1580–1626) low (see Fig. 1 b, Fig. 1 c). They are known complex functions of the frequency ϖ. The rheology properties of the layers under investigation from structural model in Fig. 1 are presented in the mathematical description (8) by corresponding layer transfer function signed W_i(jϖ) and corresponding reflection and refraction coefficients signed β_i(jϖ).

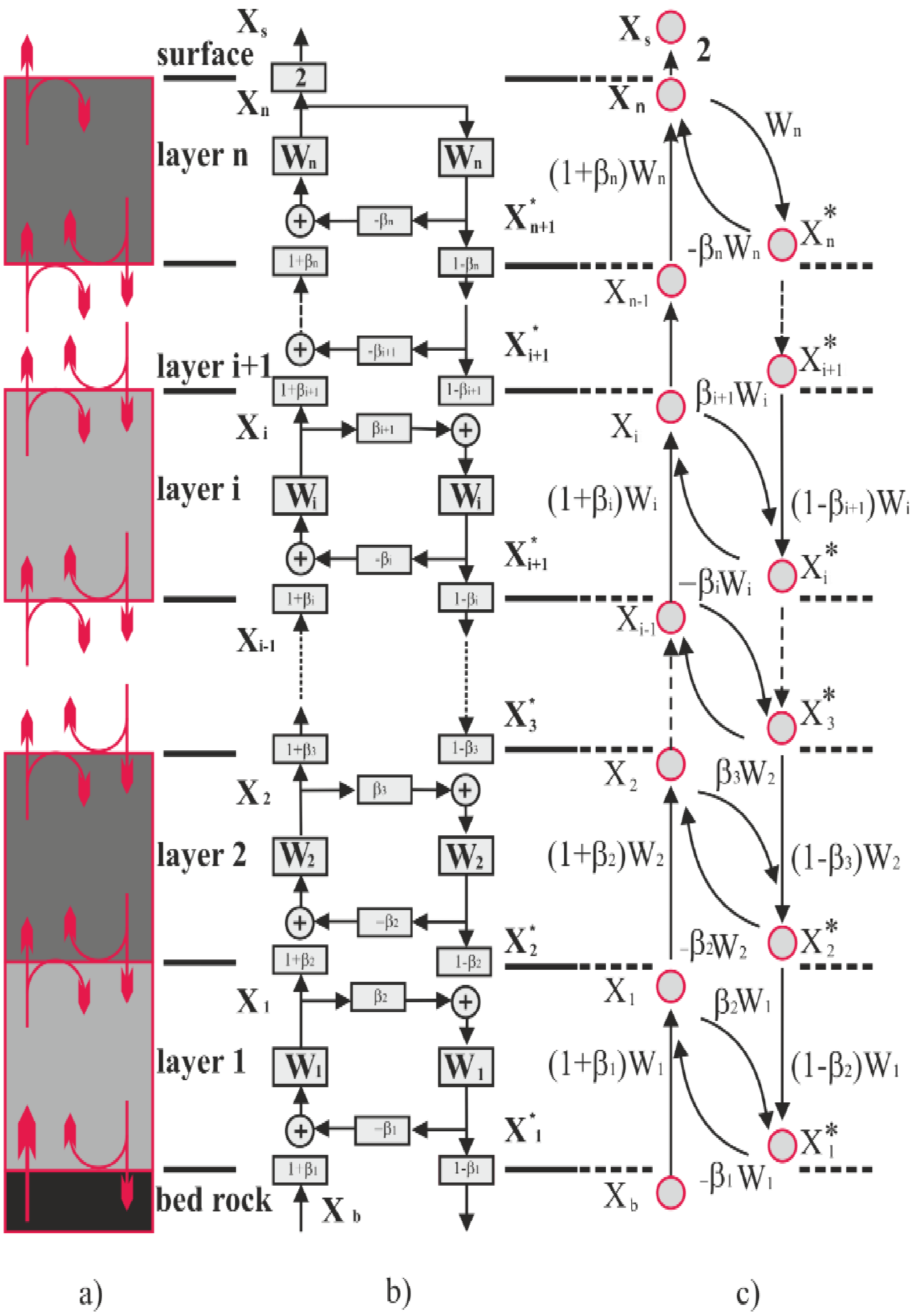

Fig. 1. Multilayered Media Structural Model. a) SH Wave Propagation Reflect – Pass
b) Perpendicular Process. c) Signal Flow Graph

By suitable selection of the real and imaginary parts of the coefficients $\beta_i(j\omega)$ can be obtained quadruple symmetric real functions presented in the first quadrant of the Fig. 2 in a capacity of searched problem solution. In case of real and imaginary parts of the coefficients β_i according to the conditions of Theorem 1 of the present paper, will get the signal in the second quadrant. In case of real and imaginary parts of the coefficients β_i according to the conditions of Theorem 3 of the previous paper, will get the signal in the third quadrant. The signals from third quadrant and from fourth quadrant can be obtained also in case of real and imaginary parts of the coefficients βi according to the conditions of Theorems 2 and 4 of the previous paper respectively. The first five theorems (they are published [8] as a sub conditions in the theorem signed by * and here points) in previous publication describes the Symmetry - Conjugation relation [1-11]:

- **Theorem 1** (The phenomenon “Symmetry” in the time domain corresponds to the phenomenon “Conjugation” in the frequency domain). The complex Fourier F(jω) spectra of the symmetric real functions in the first and second quadrants are conjugated as well as.
- **Theorem 2** The complex Fourier F(jω) spectra of the symmetric real functions in the third and fourth quadrants are conjugated respectively.
- **Theorem 3** (The phenomenon “Anti Symmetry” in the time domain corresponds to the phenomenon “Anti Conjugation” in the frequency domain). The complex Fourier F(jω) spectra of the anti symmetric real functions in the first and third quadrant are anti conjugated as well as.
- **Theorem 4.** The amplitudes of the functions in first and second quadrants are both positive, while these of the amplitudes for the functions for third and four quadrants are both negative. The functions under investigation could be of arbitrary amplitudes – negative or positive. The corresponding complex Fourier F(jω) spectra also are of arbitrary type amplitudes - negative or positive.
- **Theorem 5** (Frequency indistinguishable). Four quadruple symmetric real functions are frequency indistinguishable.
- **Theorem 6** [9,11] (The phenomenon “Symmetry” in the time domain corresponds to the phenomenon “Conjugation” in the frequency domain. The phenomenon “Anti Symmetry” in the time domain corresponds to the phenomenon “Anti Conjugation” in the frequency domain. The simultaneous operation of the Theorems 1 and 3 leads to even and odd decomposition of the Fourier complex spectrum of the common function $F^{common\, function}(j\omega)$ with length N in the time domain. This result represents spectral function, composed by the equivalent nonzero real and imaginary spectral parts with length N/2 in the frequency domain $R_{\mathrm{e}}^{even\, left}(\omega)$ and $jI_m^{odd\, right}(\omega)$) as follows:

(10) $$F^{common\, function}(j\omega) = 2\left(R_{\mathrm{e}}^{even\, left}(\omega) + jI_m^{odd\, right}(\omega)\right).$$

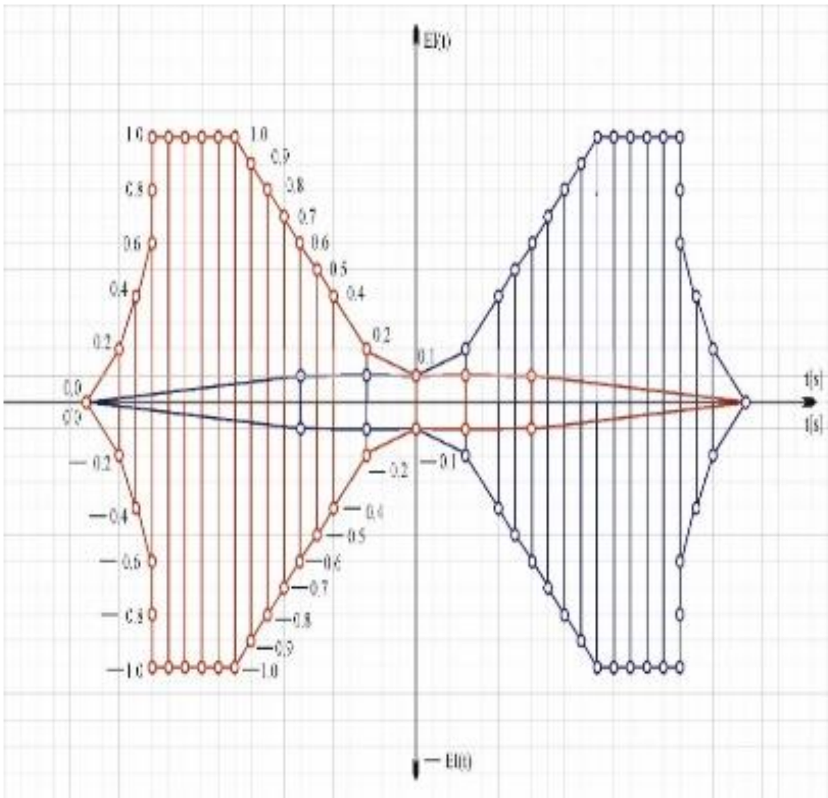

Fig. 2. Quadruple symmetric real functions

6. Conclusions. The report presents a structural analytical approach to solving wave boundary value problems - for example SH wave propagation in multi-layered structures [3,4,6,7,8,9,10,11]. The complex algebraic system (8) admits an analytic solution, since the transfer functions of the individual layers are fractional-rational functions. The transfer function of the general boundary value problem, which can be obtained from (8), is also a fractional-rational function. Such an analytical solution of wave boundary value problems in multi-layered media is proposed here for the first time. The theorems proved in the report provide an opportunity for an effective study of the obtained final integrals in the time domain of the investigated wave boundary value problems. Theorem 6 [9, 11], for example, can be interpreted as an analog variant of the Cooley and Tuckey FFT scheme [5].

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